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On boundary perturbations in Liouville theory and brane dynamics in noncritical string theories

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Abstract: We study certain relevant boundary perturbations of Liouville theory and discuss implications of our results for the brane dynamics in noncritical string theories. Our results include
(i) There exist monodromies in the parameter μ_{B} of the Neumann-type boundary condition that can create an admixture represented by the Dirichlet type boundary condition, for example.
(ii) Certain renormalization group flows can be studied perturbatively, which allows us to determine the results of the corresponding brane decays.
(iii) There exists a simple renormalization group flow that can be calculated exactly. In all the cases that we have studied the renormalization group flow acts like a covering transformation for the monodromies mentioned under (i).

Keywords: D-branes, Conformal Field Models in String Theory, Tachyon Condensation, Boundary Quantum Field Theory.

Contents

1. Introduction

In the present paper we start investigating perturbations of quantum Liouville theory by operators that break conformal invariance on the boundary of the domain on which Liouville theory is defined.

1.1 Motivation

Our motivation is three-fold:

1. There has recently been important progress concerning the role of D-branes in the dualities between noncritical string theories and matrix models, exhibiting these dualities as holographic open/closed string dualities [1]–[8]. However, the picture that is emerging from [1]–[8] does not encompass the D-branes defined by Neumann-type boundary conditions for the Liouville direction so far. The implications of the fact that one of the most basic observables of the matrix model, the macroscopic loop operator, is related to D-instantons with Neumann type boundary conditions for the Liouville direction [9] do not seem to be fully understood from the point of view of D-brane physics in noncritical string theories yet. Important clues in this direction can be expected to come from an improved understanding of the D-branes in noncritical string theories, their dynamics and their mutual relations. One of our results appears to be quite intriguing in this respect: recall that the Neumann type boundary conditions in Liouville theory are parametrized by the so-called boundary cosmological constant μ_{B}. Analytic continuation w.r.t. μ_{B} can create an admixture represented by the Dirichlet type boundary condition.[1]

2. The present understanding of time-dependent processes in string theory like the decay of D-branes does not seem to be satisfactory yet. Currently there are two main approaches for studying decay processes triggered by tachyonic modes on D-branes. In the first of these approaches one forgets about the time-direction and studies the renormalization group (RG) flows induced by perturbing the resulting boundary CFT with relevant operators. It is generally believed that the fixed points of the boundary RG flow represent possible end-points of the decay of an unstable brane, see e.g. [10] for a discussion. In a second approach to the description of brane decay processes one tries to construct an exact time dependent solution of the open string theory, i.e. a boundary CFT that involves the CFT for the time-like X^0-coordinate non-

[1] This was previously observed in [2] on the level of the boundary states. Here we show that one indeed finds complete decoupling also between the open string sectors associated to the two components.

trivially [11, 12]. We would like to point out that the noncritical string theories may serve as a useful toy model for exploring certain aspects of time-dependent processes in string theory. One has a free parameter which determines the speed of decay of the brane. Processes which take place slowly can here be studied perturbatively, unlike the examples studied in [11, 12].

3. From a somewhat more general perspective it seems important to gain some insight into the consequences of breaking conformal invariance on the boundary in conformal field theories that have a continuous spectrum. In the example that we study in this paper we will observe a remarkable new phenomenon: the couplings that correspond to generic relevant boundary fields have a trivial scale dependence only, they are "frozen". The exceptions include those boundary fields which create *normalizable* states from the vacuum. Adapting the terminology of [13] to the present situation we will call these fields "macroscopic". Perturbing with macroscopic fields will generate nontrivial RG flows which may have nontrivial infrared fixed points.

1.2 Overview

We begin our discussion in section 2 by providing the necessary results from boundary Liouville theory. Of particular importance for us will be to understand (a) the analytic properties of the theory w.r.t. the boundary parameter $\mu_{\rm B}$, and (b) the operator product expansion of boundary fields. The dependence of the theory w.r.t. $\mu_{\rm B}$ is found to be quite interesting: we will see that boundary Liouville theory lives on a certain branched cover of the complex $\mu_{\rm B}$-plane. A special case of the corresponding covering transformations turns out to be related to the remarkable relation between Dirichlet and Neumann type boundary conditions that we had mentioned above. We present and discuss the relevant results in section 2, with derivations given in the appendices.

In section 3 we discuss some examples for RG flows generated by perturbing Liouville theory with relevant boundary fields. In the case that we perturb with nearly marginal boundary fields one may use boundary perturbation theory to calculate the relevant beta-function and to determine the new fixed point that the theory flows to. We also discuss the simple soluble example of the RG flow which removes the admixture of Dirichlet-type boundary condition which is produced by performing a monodromy w.r.t. the parameter $\mu_{\rm B}$ of the Neumann type boundary conditions. In all the cases that we were able to treat we observe that the renormalization group flow acts like a covering transformation which relates different Neumann-type boundary conditions with the same value of $\mu_{\rm B}$.

The results of section 3 are interpreted from the perspective of two-dimensional string theory in section 4. The RG flows generated by the macroscopic boundary fields are interpreted in terms of brane decays that take place spontaneously, whereas the trivial scale dependence of the couplings for microscopic boundary fields is related to brane decays that are triggered by strong external sources. The latter always lead to the complete disappearance of the brane, whereas the former produces a different brane with the same value of $\mu_{\rm B}$. The soluble RG flow mentioned in the previous paragraph has an explicit time-dependent description in terms of the boundary states introduced in [11, 12].

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The appendices contain the derivations of some technical results which are interesting in their own right. This includes in particular a detailed study of the analytic properties of the structure functions, and an analysis of the operator product expansion of boundary fields in boundary Liouville theory.

1.3 Conventions

We will assume that the reader is familiar with Liouville theory in the case of periodic boundary conditions on the cylinder, as discussed in [14] and references therein. Our conventions will follow those of [15, 14], with the following exception. Let $|\alpha\rangle$ be the state generated by acting with the primary field $V_\alpha(z,\bar{z})$ on the vacuum,

$$|\alpha\rangle := \lim_{z\to 0} V_\alpha(z,\bar{z})|0\rangle\,. \tag{1.1}$$

We then define $|P\rangle$ by

$$|P\rangle = \frac{1}{\sqrt{2\pi}}\,|\alpha\rangle\,, \qquad \text{if } \alpha = \frac{Q}{2} + iP\,, \tag{1.2}$$

which is normalized such that $\langle P'|P\rangle = \delta(P'-P)$ if $P, P' \in \mathbb{R}$.

2. More on boundary Liouville theory

As a preparation we will need to develop our understanding of quantum Liouville theory with conformally invariant boundary conditions [9, 16, 17, 18] a little further. To be specific, let us consider Liouville theory on the upper half plane, with certain boundary conditions imposed along the real axis. For the reader's convenience let us begin by reviewing the necessary results from [9, 16, 17, 18].

2.1 Boundary conditions

Liouville theory permits two types of boundary conditions which preserve conformal invariance: one being of Dirichlet type [17], the other being a generalization of Neumann type boundary conditions [9, 16, 18].

The Dirichlet type boundary condition is classically defined by the requirement that the classical Liouville field $\varphi(z,\bar{z})$ diverges near the real axis like $-2\log \mathrm{Im}(z)$. The corresponding boundary condition for quantum Liouville theory may be characterized by the boundary state $\langle B_{\mathrm{D}}|$, which we will write as

$$\langle B_{\mathrm{D}}| = \int_0^\infty dP\,\Psi_{\mathrm{ZZ}}(P)\,\langle\!\langle P|\,, \qquad \Psi_{\mathrm{ZZ}}(P) = -2^{\frac{3}{4}}\,v(P)\,, \tag{2.1}$$

where $|P\rangle\!\rangle$ is the Ishibashi-state built from $|P\rangle$, and the function $v(P)$ is defined by

$$v(P) := (\pi\mu\gamma(b^2))^{-\frac{iP}{b}}\,\frac{-4\pi iP}{\Gamma(1-2ibP)\Gamma(1-2iP/b)}\,. \tag{2.2}$$

The second important class of conformally invariant boundary conditions for Liouville theory may be defined in the semi-classical limit $c \to \infty$ by imposing the boundary condition

$$i(\partial - \bar{\partial})\, \varphi = 4\pi\mu_{\rm B} b^2\, e^{\frac{\varphi}{2}}\,. \tag{2.3}$$

along the real axis. The parameter $\mu_{\rm B}$ which labels the boundary conditions is called the boundary cosmological constant. The corresponding boundary states were constructed in [9]. They can be represented as

$$\langle B_\sigma | = \frac{1}{2\pi}\int_{\mathbb{S}} d\alpha\; A_\alpha^\sigma\; \langle\!\langle \alpha |\,, \tag{2.4}$$

where $\mathbb{S} = \frac{Q}{2} + i\mathbb{R}^+$, the parameter σ used in (2.4) is related to the boundary cosmological constant $\mu_{\rm B}$ via

$$\cos \pi b(2\sigma - Q) = \frac{\mu_{\rm B}}{\sqrt{\mu}}\sqrt{\sin \pi b^2}\,, \tag{2.5}$$

and the one-point function A_α^σ is given as

$$A_\alpha^\sigma = \sqrt{\frac{\pi}{2}}\frac{\cosh(2\pi P(2\sigma - Q))}{\sinh 2\pi b P \sinh 2\pi b^{-1} P}\Psi_{\rm ZZ}(P) \qquad \text{if } \alpha = \frac{Q}{2} + iP\,. \tag{2.6}$$

We will see later that correlation functions of boundary Liouville theory have nice analytic properties in their dependence on the boundary parameter σ. The relation (2.5) uniformizes the branched cover of the complex $\mu_{\rm B}$-plane on which boundary Liouville theory can be defined. To begin with, we will consider the case where the parameter σ takes values in $\mathbb{S}$. We notice that we then have a one-to-one correspondence between the boundary conditions labelled by $\sigma \in \mathbb{S} := \frac{Q}{2} + i\mathbb{R}^+$ and the spectrum of Liouville theory with periodic boundary conditions ("Cardy-case").

2.2 Spectrum I — Continuous part

Let us consider Liouville theory on the strip $[0, \pi]$ with boundary conditions labelled by parameters σ_2 and σ_1. The spectrum $\mathcal{H}^{\rm B}_{\sigma_2\sigma_1}$ of Liouville theory on the strip has been determined in [16]. It may be represented as

$$\mathcal{H}^{\rm B}_{\sigma_2\sigma_1} = \int_{\mathbb{S}} d\alpha\, \mathcal{V}_{\alpha,c}\,, \tag{2.7}$$

where $\mathbb{S} = \frac{Q}{2} + i\mathbb{R}^+$ and $\mathcal{V}_{\alpha,c}$ is the highest weight representation of the Virasoro algebra with weight $\Delta_\alpha = \alpha(Q - \alpha)$ and central charge $c = 1 + 6Q^2$. We will denote by $v_\alpha^{\sigma_2\sigma_1}$ the representative of the highest weight state of $\mathcal{V}_{\alpha,c}$ in $\mathcal{H}^{\rm B}_{\sigma_2\sigma_1}$, normalized by

$$\langle\, v_{\alpha_2}^{\sigma_2\sigma_1}\,,\, v_{\alpha_1}^{\sigma_2\sigma_1}\,\rangle_{\mathcal{H}^{\rm B}_{\sigma_2\sigma_1}} = \delta_{\mathbb{S}}(\alpha_2 - \alpha_1)\,. \tag{2.8}$$

It is sometimes useful to note [16] that in the weak coupling asymptotics $\phi \to -\infty$ one may describe the states $v_\alpha^{\sigma_2\sigma_1}$ by the wave-functions $\psi_\alpha^{\sigma_2\sigma_1}(q)$, $q := \int_0^\pi dx \varphi(x)$, such that

$$\psi_\alpha^{\sigma_2\sigma_1}(q)\,\sim\,\frac{1}{\sqrt{2\pi}}e^{-\frac{Q}{2}q}(e^{\alpha q} + r_\alpha^{\sigma_2\sigma_1} e^{(Q-\alpha)q}) \qquad \text{for } q \to -\infty\,, \tag{2.9}$$

where the reflection amplitude $r_\alpha^{\sigma_2,\sigma_1}$ has an explicit expression given by

$$r_\alpha^{\sigma_2\sigma_1} = (\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{Q-2\alpha}{2b}} \frac{\Gamma_b(2\alpha-Q)}{\Gamma_b(Q-2\alpha)} \frac{S_b(2Q-\sigma_2-\sigma_1-\alpha)S_b(\sigma_2+\sigma_1-\alpha)}{S_b(\alpha+\sigma_1-\sigma_2)S_b(\alpha+\sigma_2-\sigma_1)} . \tag{2.10}$$

Definitions and relevant properties of the special functions Γ_b and S_b are reviewed in the appendix A.1. For the moment let us simply note that $\Gamma_b(x)$ and $S_b(x)$ are analytic in the strip $0 < \mathrm{Re}(x) < Q$ and have a simple pole at $x = 0$.

2.3 Boundary fields

The states $v_P^{\sigma_2\sigma_1}$ are created from the vacuum by the boundary fields $\Psi_\alpha^{\sigma_2\sigma_1}(x)$, $\alpha = \frac{Q}{2}+iP$. These boundary fields are fully characterized by the three point function

$$\begin{aligned}\langle \Psi_{\alpha_3}^{\sigma_1\sigma_3}(x_3)\Psi_{\alpha_2}^{\sigma_3\sigma_2}(x_2)\Psi_{\alpha_1}^{\sigma_2\sigma_1}(x_1)\rangle = {} & |x_3-x_2|^{\Delta_{\alpha_1}-\Delta_{\alpha_3}-\Delta_{\alpha_2}}|x_3-x_1|^{\Delta_{\alpha_2}-\Delta_{\alpha_3}-\Delta_{\alpha_1}} \times \\ & \times |x_2-x_1|^{\Delta_{\alpha_3}-\Delta_{\alpha_2}-\Delta_{\alpha_1}} C_{\alpha_3\alpha_2\alpha_1}^{\sigma_3\sigma_2\sigma_1}\end{aligned} \tag{2.11}$$

The explicit formula for $C_{\alpha_3\alpha_2\alpha_1}^{\sigma_3\sigma_2\sigma_1} \equiv C_{Q-\alpha_3|\alpha_2\alpha_1}^{\sigma_3\sigma_2\sigma_1}$ found in [18] is reviewed and further studied in appendix A.

An explicit description for the operator product expansion of boundary operators can be found by combining the results of [18] and [19]:

$$\Psi_{\alpha_2}^{\sigma_3\sigma_2}(x_2)\Psi_{\alpha_1}^{\sigma_2\sigma_1}(x_1) = \int_{\mathbb{S}} d\alpha_3 C_{\alpha_3|\alpha_2\alpha_1}^{\sigma_3\sigma_2\sigma_1} |x_2-x_1|^{\Delta_{\alpha_3}-\Delta_{\alpha_2}-\Delta_{\alpha_1}} \Psi_{\alpha_3}^{\sigma_3\sigma_1}(x_1) + \text{descendants} , \tag{2.12}$$

The expansion (2.12) is applicable if $|\,\mathrm{Re}(\alpha_1-\alpha_2)| < Q/2$ and $|\,\mathrm{Re}(\alpha_1+\alpha_2-Q)| < Q/2$. The general case can be obtained from (2.12) by means of analytic continuation, see appendix C for more details.

For these and other purposes it turns out to be rather useful to change the normalization of the boundary fields via

$$\Phi_\alpha^{\sigma_2\sigma_1}(x) = g_\alpha^{\sigma_2\sigma_1}\Psi_\alpha^{\sigma_2\sigma_1}(x) , \tag{2.13}$$

where the function $g_\alpha^{\sigma_2\sigma_1}$ will be chosen as [18]

$$g_\alpha^{\sigma_2\sigma_1} = \frac{(\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{\alpha}{2b}}}{\Gamma_b(2Q-\alpha-\sigma_1-\sigma_2)} \frac{\Gamma_b(Q)\Gamma_b(Q-2\alpha)\Gamma_b(2\sigma_1)\Gamma_b(2Q-2\sigma_2)}{\Gamma_b(Q-\alpha+\sigma_1-\sigma_2)\Gamma_b(Q-\alpha+\sigma_2-\sigma_1)\Gamma_b(\sigma_1+\sigma_2-\alpha)} . \tag{2.14}$$

Using the fields $\Phi_\alpha^{\sigma_2\sigma_1}(x)$ instead of $\Psi_\alpha^{\sigma_2\sigma_1}(x)$ has considerable advantages, for example:

1. The fields $\Phi_\alpha^{\sigma_2\sigma_1}(x)$ have a particularly simple behavior under the reflection $\alpha \to Q-\alpha$, namely

   $$\Phi_\alpha^{\sigma_2\sigma_1}(x) = \Phi_{Q-\alpha}^{\sigma_2\sigma_1}(x) . \tag{2.15}$$

   This implies in particular that $\Phi_\alpha^{\sigma_2\sigma_1}(x)$ is non-vanishing at $\alpha{=}Q/2$ (unlike $\Psi_\alpha^{\sigma_2\sigma_1}(x)$). In the case $c = 25 \Leftrightarrow b = 1$ this implies furthermore that the operator that represents the boundary interaction can be identified with $\Phi_1^{\sigma,\sigma}(x)$, see [4, section 2].

2. The dependence of $\Phi_\alpha^{\sigma_2\sigma_1}(x)$ w.r.t. the variable α is not only *meromorphic*, it is analytic for $0 \leq \mathrm{Re}(\alpha) \leq Q$. This means in particular that correlation functions like

$$\left\langle \prod_{i=1}^{n} \Phi_{\alpha_i}^{\sigma_{i+1}\sigma_i}(x_i) \right\rangle \tag{2.16}$$

are meromorphic w.r.t. each variable α_i, with poles that can be avoided by a variation of or by smearing over the remaining variables α_j, $j \neq i$.

While the first of these statements is easily checked, it is nontrivial to prove the second. The main ingredients of the proof are contained in the appendices A and C.

2.4 Analyticity in μ_{B}

Much of the following will rely on the fact that boundary Liouville depends *analytically* on the boundary parameter σ introduced in (2.5). In order to exhibit the analytic properties w.r.t. σ more clearly let us introduce yet another class of boundary fields as

$$\tilde{\Phi}_\alpha^{\sigma_2\sigma_1}(x) = G_\alpha^{\sigma_2\sigma_1}\Psi_\alpha^{\sigma_2\sigma_1}(x)\,, \tag{2.17}$$

where $G_\alpha^{\sigma_2\sigma_1}$ is related to $g_\alpha^{\sigma_2\sigma_1}$ by canceling the factors that depend on σ_1, σ_2 only,

$$G_\alpha^{\sigma_2\sigma_1} = \frac{\Gamma_b^2(Q)}{\Gamma_b(2\sigma_1)\Gamma_b(2Q-2\sigma_2)} g_\alpha^{\sigma_2\sigma_1}\,. \tag{2.18}$$

The analytic properties of the corresponding structure functions are analyzed in the appendices A–C. These results imply that

> The correlation functions
> $\langle V_{\alpha_n}(z_n,\bar{z}_n)\ldots V_{\alpha_1}(z_1,\bar{z}_1)\tilde{\Phi}_{\beta_m}^{\sigma_1\sigma_m}(x_m)\ldots\tilde{\Phi}_{\beta_1}^{\sigma_2\sigma_1}(x_1)\rangle$
> are entire analytic w.r.t.$\sigma_1,\ldots,\sigma_m$.

With the help of eq. (2.5) one may translate analyticity w.r.t. the boundary parameter σ back into a statement about the analytic properties of boundary Liouville theory w.r.t. μ_{B}. One should note that the structure functions do not share the periodicity of μ_{B} under $\sigma \to \sigma + b^{-1}$, which means that the theory exhibits *monodromies* if one considers the analytic continuation w.r.t. μ_{B} along closed paths. This may be described by introducing two branch cuts into the complex μ_{B}-plane, one running from $-\infty$ to $-\sqrt{\frac{\mu}{\sin \pi b^2}}$ and one between $-\sqrt{\frac{\mu}{\sin \pi b^2}}$ and $\sqrt{\frac{\mu}{\sin \pi b^2}}$. We will later discuss a particularly interesting example of such a monodromy.

2.5 The spectrum II — Bound states

We shall now discuss the case where $\sigma \in \mathbb{R}$. To simplify the discussion slightly we will focus on the cases where the boundary conditions σ_2 and σ_1 to the left and right of the insertion point of a boundary operator are always equal, $\sigma_2 = \sigma_1 \equiv \sigma$. The corresponding boundary operators will be denoted as $\Psi_\alpha^\sigma(x)$. We will furthermore impose the “Seiberg-bound” $\sigma < Q/2$ throughout.

The spectrum now shows an interesting dependence w.r.t. σ. It remains unchanged as long as $Q/4 < \sigma < Q/2$. Of primary interest for us will be the case that $\sigma < Q/4$, where one finds in addition to (2.7) a discrete part in the spectrum [16]:

$$\mathcal{H}^{\mathrm{B}}_{\sigma\sigma} = \int_{\mathbb{S}} d\alpha\, \mathcal{V}_{\alpha,c} \oplus \bigoplus_{\alpha\in\mathbb{D}_s} \mathcal{V}_{\alpha,c}\,, \tag{2.19}$$

where $\mathbb{D}_s = \{\alpha \in \mathbb{C}; \alpha = 2\sigma + nb + mb^{-1}\,,\, n,m \in \mathbb{Z}^{\geq 0}\}$. For $\sigma < 0$ one finds non-unitary representations of the Virasoro algebra, which is why we will mostly discuss $0 \leq \sigma < Q/4$ in the following. In the case of our main interest, $b = 1$, we find only a single bound state at $\alpha = 2\sigma$ as long as $\sigma > 0$.

In order to get an alternative point of view on the origin of a discrete part in the spectrum let us note that the operators $\Phi^\sigma_\alpha(x)$ create states $w_{\alpha;\sigma}$ with asymptotic wave-functions $\phi^\sigma_\alpha(q)$, $q := \int_0^\pi dx\varphi(x)$, such that

$$\phi^\sigma_\alpha(q) \sim \frac{1}{\sqrt{2\pi}}\, e^{-\frac{Q}{2}q}(g^\sigma_\alpha\, e^{\alpha q} + g^\sigma_{Q-\alpha}\, e^{(Q-\alpha)q}) \qquad \text{for } q \to -\infty\,, \tag{2.20}$$

The asymptotics (2.20) implies that the wave-functions $\phi^\sigma_\alpha(q)$ will be generically non-normalizable for $\mathrm{Im}(\alpha) \neq Q/2$. However, the zero of g^σ_α at $\alpha = 2\sigma$ suggests that $w_\sigma := w_{2\sigma;\sigma}$ represents a *normalizable* state.

It can indeed be shown directly that the corresponding vertex operator $\Phi_\sigma(x) := \Phi^\sigma_{2\sigma}(x)$ creates normalizable states: the two-point-function of the operator $\Phi^\sigma_\alpha(x)$ can be constructed as

$$d^\sigma_\alpha\, |x_2 - x_1|^{-2\Delta_{2\sigma}} := \langle \Phi^\sigma_\alpha(x_2)\Phi^\sigma_\alpha(x_1)\rangle = \lim_{\alpha_3\to 0} \langle \Phi^\sigma_{\alpha_3}(x_3)\Phi^\sigma_\alpha(x_2)\Phi^\sigma_\alpha(x_1)\rangle\,. \tag{2.21}$$

This allows us to calculate d^σ_α from the OPE-coefficients $C^{\sigma_3\sigma_2\sigma_1}_{\alpha_3\alpha_2\alpha_1}$. The result is infinite (proportional to $\delta(0)$) for $\alpha \neq 2\sigma$, but turns out to be *finite* if $\alpha = 2\sigma$. It is then given by the expression

$$\begin{aligned} d_\sigma &:= (\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{Q}{2b}} \frac{\Gamma_b(Q-4\sigma)\Gamma_b^2(2Q-2\sigma)}{\Gamma_b(2Q-4\sigma)\Gamma_b^2(Q-2\sigma)} \\ &= 2\pi(\pi\mu\gamma(b^2))^{\frac{Q}{2b}} \frac{Q-4\sigma}{(Q-2\sigma)^2} \frac{\Gamma(b(Q-4\sigma))\Gamma(b^{-1}(Q-4\sigma))}{\Gamma^2(b(Q-2\sigma))\Gamma^2(b^{-1}(Q-2\sigma))}\,. \end{aligned} \tag{2.22}$$

Let us furthermore note that the operator product expansion (2.12) has an analytic continuation to more general values of $\sigma_1, \sigma_2, \sigma_3$ and α_1, α_2, see the appendix C for a more detailed discussion. Here we are interested in the case that $\sigma_i = \sigma < Q/4$, $i = 1,2,3$. The first important point to observe is that in the analytically continued OPE one will always find a discrete contribution proportional to Φ_σ, which was to be expected due to the appearance of w_σ in the discrete part of the spectrum $\mathcal{H}^{\mathrm{B}}_{\sigma,\sigma}$.

In the case that will be our main focus later, namely $2b^2 > 1$ and $Q < 2\,\mathrm{Re}(\alpha_1 + \alpha_2) < 2Q$, it turns out that the contribution proportional to Φ_σ is the only one that appears discretely. The operator product expansion for $\Phi^{\sigma}_{\alpha_2}(x_2)\Phi^{\sigma}_{\alpha_1}(x_1)$ then takes the following form:

$$\begin{aligned}\Phi^{\sigma}_{\alpha_2}(x_2)\Phi^{\sigma}_{\alpha_1}(x_1) = & \int_{\mathbb{S}} d\alpha_3\, E^{\sigma}_{\alpha_3|\alpha_2\alpha_1} |x_2 - x_1|^{\Delta_{\alpha_3}-\Delta_{\alpha_2}-\Delta_{\alpha_1}} \Phi^{\sigma}_{\alpha_3}(x_1) + \\ & + E^{\sigma}_{\alpha_2\alpha_1} |x_2 - x_1|^{-\Delta_{2\sigma}} \Phi^{\sigma}_{2\sigma}(x_1) + \text{descendants}\,.\end{aligned} \tag{2.23}$$

Of particular importance for us will be the operator product coefficient $E_\sigma := E^{\sigma}_{2\sigma,2\sigma}$, which has the explicit expression

$$E_\sigma = \frac{\Gamma_b(2Q-6\sigma)}{\Gamma_b(Q)} \frac{\Gamma_b(2Q-2\sigma)}{\Gamma_b(Q-4\sigma)} \frac{\Gamma_b^2(Q-2\sigma)}{\Gamma_b^2(2Q-4\sigma)}\,. \tag{2.24}$$

The derivation of this expression is given in the appendix A.4.

2.6 The limit $\sigma \to 0$

A particularly interesting value for the boundary parameter σ turns out to be $\sigma = 0$. To begin with, let us note [2] that

$$\langle B_0| = \langle B_b| + \langle B_{\rm D}|\,, \tag{2.25}$$

where $\langle B_{\rm D}| = \int dP\, \Psi_{ZZ}\langle P|$ is the boundary state introduced in (2.1). Let us next note that the spectrum of boundary Liouville theory at $\sigma = 0$ contains, in particular, the vacuum representation $\mathcal{V}_{0,c}$, as follows either from the Cardy-type computation in [16], or by using the decomposition (2.25) together with formula (5.9) from [17].

These observations suggest that one may view the boundary Liouville theory with $\sigma = 0$ as a kind of superposition of the boundary Liouville theory with $\sigma = b$ and the theory corresponding to the Dirichlet type boundary conditions from [17]. This expectation turns out to be realized in a rather accurate sense. In the rest of this section we will summarize some interesting features of boundary Liouville theory at $\sigma = 0$ which are derived in the appendix D.

The vertex operator Φ_0 that corresponds to the highest weight state in $\mathcal{V}_{0,c}$ can be constructed by taking the limit $\sigma \to 0$ of the boundary field $\Phi_\sigma(x)$. It is shown in appendix D that

$$\boxed{\Phi_0 = \lim_{\sigma\to 0} \Phi_\sigma(x) = \Pi_0\,,} \tag{2.26}$$

where Π_0 denotes the *projection* onto the subspace $\mathcal{V}_{0,c}$ in $\mathcal{H}^{\rm B}_{0,0}$.

In order to construct the fields that create the states in the complement $\mathcal{V}^{\perp}_{0,c}$ of $\mathcal{V}_{0,c}$ one needs to consider the boundary fields $\tilde{\Phi}^{\sigma\sigma}_{\alpha}(x)$ instead of $\Phi^{\sigma\sigma}_{\alpha}(x)$, cf. (2.17). We define $\tilde{\Phi}_\alpha(x) := \lim_{\sigma\to 0} \tilde{\Phi}^{\sigma\sigma}_{\alpha}(x)$. The sectors $\mathcal{V}_{0,c}$ and $\mathcal{V}^{\perp}_{0,c}$ turn out to be completely decoupled, in the sense that all mixed correlation functions which contain fields of both types Φ_0 and $\tilde{\Phi}_\alpha$ vanish.

The boundary field Φ_0 acts as a projector in yet another way. We have

$$\boxed{\begin{aligned}\langle B_0|\Phi_0\, V_{\alpha_n}(z_n,\bar{z}_n)\, &\ldots V_{\alpha_1}(z_1,\bar{z}_1)|0\rangle = \\ &= \langle\, B_{\rm D}|V_{\alpha_n}(z_n,\bar{z}_n)\ldots V_{\alpha_1}(z_1,\bar{z}_1)|0\rangle\,.\end{aligned}} \tag{2.27}$$

This means that inserting Φ_0 on the boundary of a disk with the $\sigma = 0$ boundary condition projects out the couplings to the term $\langle B_b|$ in (2.25). Equation (2.27) encodes a rather remarkable relation between the two different types of boundary conditions in Liouville theory.

Let us finally remark that the analytic continuation w.r.t. σ from $\sigma = b$ to $\sigma = 0$ corresponds to analytically continuing w.r.t. $\mu_{\rm B}$ along a closed cycle starting from

$$\mu_{\rm B} = -\sqrt{\frac{\mu}{\sin \pi b^2}} \cos \pi b^2 ,$$

and returning to the same value for $\mu_{\rm B}$. Performing the analytic continuation along such a monodromy cycle creates an admixture represented by $\langle B_{\rm D}|$.

3. Boundary perturbations

Our aim will be to study the renormalization group flow generated by boundary perturbations of the following form

$$S_{\rm pert} := -\lambda_\alpha a^{-y_\alpha} \int_{\partial\Sigma} dx\, \Phi^\sigma_\alpha(x) , \tag{3.1}$$

where we will mostly assume that the parameter $y_\alpha := 1 - \Delta_\alpha$ satisfies $1 \gg y_\alpha > 0$. Our discussion will partially follow the treatment of similar problems in [20, 21, 22], concentrating onto the new features that originate from the continuous spectrum of boundary Liouville theory.

3.1 Boundary renormalization group flows

The correlation functions in the perturbed theory are formally defined by

$$\langle \mathcal{O} \rangle_{\sigma;\lambda_\alpha} := \langle \mathcal{O} {\rm Pexp}(-S_{\rm pert}) \,\rangle_\sigma , \tag{3.2}$$

where $\mathcal{O}$ represents the operator insertions that define the correlation function in question, and Pexp is the path ordered exponential.

Turning on the perturbation (3.1) will of course spoil scale invariance. Let us check explicitly that $T(z) \neq \bar{T}(\bar{z})$ when $\lambda \neq 0$. We will temporarily adopt the choice $\Sigma = \mathbb{H}^+$, the upper half plane. To first order in λ we have to consider

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \int_{\partial\Sigma} dy \Big\langle (T(x+i\epsilon) - \bar{T}(x-i\epsilon)) \Phi^\sigma_\alpha(y) \mathcal{O} \Big\rangle = & \\ = \lim_{\epsilon \downarrow 0} \int_{\partial\Sigma} dy \Big\langle \Big[\Big(\frac{\Delta_\alpha}{(x-y+i\epsilon)^2} - \frac{\Delta_\alpha}{(x-y-i\epsilon)^2} \Big) \Phi^\sigma_\alpha(x) + & \\ + \Big(\frac{1-\Delta_\alpha}{x-y+i\epsilon} - \frac{1-\Delta_\alpha}{x-y-i\epsilon} \Big) \partial_x \Phi^\sigma_\alpha(x) \Big] \mathcal{O} \Big\rangle & \\ = 2\pi(\Delta_\alpha - 1) \langle \partial_x \Phi^\sigma_\alpha(x) \mathcal{O} \rangle \, . & \end{aligned} \tag{3.3}$$

This can be seen as an analogy to the well-known fact that in the case of bulk perturbations the trace of the energy-momentum tensor is proportional to the perturbing field itself, see e.g. [20, section 6.1].

A useful tool for describing the scale-dependence of the perturbed theory is the renormalization group (RG). Let us consider the effective action defined by choosing $\Sigma = D_L$, a disk with circumference $2\pi L$, and by introducing the ultraviolet cut-off $|x_i - x_j| < a$ in the integrals that one gets in the perturbative expansion of (3.2). In order to make the effective action independent of the choice of cut-off we will have to compensate the result of a change of a by a corresponding change of the bare coupling constant λ_α. The coupling constant λ_α will thereby become dependent on the dimensionless scale-parameter $l = \ln(a/L)$. To first order in λ we find from the explicit scale dependence in (3.1) that a variation $\delta_\epsilon : a \to a(1+\epsilon)$ of the cut-off must be compensated by a variation $\delta_\epsilon \lambda_\alpha = y\lambda_\alpha \epsilon + \mathcal{O}(\lambda_\alpha^2)$.

In order to find the contribution to $\delta_\epsilon \lambda$ of order $\mathcal{O}(\lambda^2)$ we have to calculate

$$\begin{aligned} \delta_\epsilon^{(2)} &:= \delta_\epsilon \left(\frac{1}{2} \int_{\partial\Sigma} d\varphi_2 d\varphi_1 \, \Phi_\alpha^\sigma(\varphi_2) \Phi_\alpha^\sigma(\varphi_1) \Theta(|\varphi_2 - \varphi_1| - e^l) \right) \\ &= -\frac{\epsilon}{2} \int_{\partial\Sigma} d\varphi_2 d\varphi_1 \, \Phi_\alpha^\sigma(\varphi_2) \Phi_\alpha^\sigma(\varphi_1) \delta(|\varphi_2 - \varphi_1| - e^l) \,, \end{aligned} \tag{3.4}$$

where we have parametrized ∂D_L by coordinates φ_i, $i = 1,2$. For small a/L we may use the OPE of $\Phi_\alpha^\sigma(\varphi_2)\Phi_\alpha^\sigma(\varphi_1)$ in order to further evaluate (3.4). To keep the discussion simple let us assume that $2b^2 > 1$. The condition that $\Phi_\alpha^\sigma(x)$ defines a nearly marginal boundary perturbation then requires that $Q/4 < \alpha < b$. The OPE (2.23) is applicable under these conditions and takes the simple form

$$\Phi_\alpha^\sigma(\varphi_2)\Phi_\alpha^\sigma(\varphi_1) = E_{\alpha\alpha}^\sigma |\varphi_2 - \varphi_1|^{\Delta_{2\sigma} - 2\Delta(\alpha)} \Phi_{2\sigma}^\sigma(\varphi_1) + \text{irrelevant fields} \,. \tag{3.5}$$

Inserting (3.5) into (3.4) yields the term

$$\delta_\epsilon^{(2)} = -\epsilon \, E_{\alpha\alpha}^\sigma \left(\frac{a}{L}\right)^{\Delta_{2\sigma} - 2\Delta(\alpha)} \int_{\partial\Sigma} d\varphi_1 \; \Phi_{2\sigma}(\varphi_1), \tag{3.6}$$

which has to be cancelled by a corresponding variation of λ_α with the opposite sign. We clearly have to distinguish two cases:

1. Let $\alpha \neq 2\sigma$. Eq. (3.6) implies that λ_α shows no scale dependence in second order of perturbation theory. It is rather clear that this will also be found in higher orders of boundary perturbation theory, the reason being simply the absence of Φ_α^σ in the OPE for $\alpha \neq 2\sigma$. On the other hand we find that an additional perturbation by (3.1) with $\alpha = 2\sigma$ is generated at $\mathcal{O}(\lambda_\alpha^2)$.

2. Let $\alpha = 2\sigma$. We then find that the scale variation of $\lambda := \lambda_{2\sigma}$ is given by

$$\delta_\epsilon \lambda = \epsilon(y\lambda + E_\sigma \lambda^2) + \mathcal{O}(\lambda^3) \,, \quad \text{with } E_\sigma := E_{2\sigma,2\sigma}^\sigma \,. \tag{3.7}$$

We have arrived at a remarkable conclusion: only the coupling λ that corresponds to the field which creates the *normalizable* state w_σ in the boundary Liouville theory shows a nontrivial scale-dependence, all others are "frozen".

3.2 Determination of the new fixed point

The dependence of the coupling λ w.r.t. the scale-parameter $l = \ln(L/\epsilon)$ is determined by the flow-equation that follows from (3.7). Ignoring terms of $\mathcal{O}(y)$, where $y := y_{2\sigma}$ we find

$$\dot{\lambda} := \frac{d\lambda}{dl} = \lambda y + E_\sigma \lambda^2 + \mathcal{O}(\lambda^3)\,, \quad \text{with } E_\sigma := E^\sigma_{2\sigma,2\sigma} \tag{3.8}$$

For small values of y one therefore finds a nontrivial fixed point of the renormalization group flow, $\beta(\lambda_*) = 0$, at

$$\lambda^* = -\frac{y}{E_\sigma} + \mathcal{O}(y^2)\,. \tag{3.9}$$

For consistency we need to make sure that the condition that the perturbation is nearly marginal, $0 < y \ll 1$ really implies that λ^* is small. It will be convenient to use $\epsilon := b - 2\sigma$ as the small parameter from now on. For $b \neq 1$ one has $y = \mathcal{O}(\epsilon)$, and E_σ is analytic for 2σ near b, so that $\lambda^* = \mathcal{O}(\epsilon)$. In the case $b = 1$ we find that $y = \mathcal{O}(\epsilon^2)$, but $E_\sigma = \mathcal{O}(\epsilon)$ due to the factor $1/\Gamma_b(2 - 4\sigma)$ in (2.24), so that again $\lambda^* = \mathcal{O}(\epsilon)$.

It remains to describe the new fixed point at $\lambda = \lambda^*$ in terms of the known boundary conditions labelled by σ. We may observe that $\mu_{\rm B}$ is the coefficient of the operator $\int_{\partial\Sigma} dx \Psi_b^{\sigma\sigma}(x)$ which creates *non-normalizable* states. The perturbing operator $\int_{\partial\Sigma} dx \Phi_\sigma(x)$, on the other hand, creates *normalizable* states. This strongly indicates that the theory at the fixed point can not have a value of $\mu_{\rm B}$ which differs from the unperturbed theory.[2] The most natural candidate for the boundary parameter σ_* at the new fixed point is therefore

$$\sigma_* = \frac{b}{2} + \frac{\epsilon}{2}\,, \qquad \text{given that} \qquad \sigma = \frac{b}{2} - \frac{\epsilon}{2}\,. \tag{3.10}$$

σ_* and σ correspond to the same value of $\mu_{\rm B}$, but parameterize points on different sheets of the branched covering of the $\mu_{\rm B}$-plane that boundary Liouville theory lives on.

In order to verify that (3.10) indeed holds, let us consider the leading order correction δA^σ_α to the one point function A^σ_α which is given by the expression

$$\begin{aligned} \delta A^\sigma_\alpha &= \lambda^* a^{-y} \int_{\partial\Sigma} dx\, \langle B_\sigma|\, \Phi_\sigma(x) V_\alpha(0)|0\rangle \\ &= 2\pi\lambda^* \langle B_{\frac{b}{2}}|\Phi_{\frac{b}{2}}(1) V_\alpha(0)|0\rangle + \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{3.11}$$

The explicit expression for the bulk-boundary two-point function in (3.11) is derived in appendix B.3. It is given by the expression

$$\langle B_{\frac{b}{2}}|\Phi_{\frac{b}{2}}(1) V_\alpha(0)|0\rangle = \frac{1}{\sqrt{2\pi}} \frac{2\pi b P}{\sinh 2\pi b P} \Psi_{\rm ZZ}(P)\,. \tag{3.12}$$

Let us furthermore observe that the expression for λ^* may be simplified as

$$\lambda^* = -y E_\sigma^{-1} = -\epsilon b^{-1} + \mathcal{O}(\epsilon^2)\,. \tag{3.13}$$

[2] I would like to thank J. Maldacena for drawing my attention to this fact, which led me to correct an important error in a previous version of this paper.

By inserting (3.12) and (3.13) into (3.11) we finally arrive at the expression

$$\delta A_{\alpha}^{\sigma} = \epsilon\sqrt{\frac{\pi}{2}}\,\frac{-4\pi P}{\sinh 2\pi bP}\Psi_{\rm ZZ}(P)\,. \tag{3.14}$$

In order to verify (3.10) it remains to observe that $A_{\alpha}^{\sigma} + \delta A_{\alpha}^{\sigma}$ as given by (3.14) may also be written as

$$A_{\alpha}^{\sigma} + \delta A_{\alpha}^{\sigma} = A_{\alpha}^{\sigma} + \epsilon\left[\frac{\partial}{\partial\sigma}A_{\alpha}^{\sigma}\right]_{\sigma=\frac{b}{2}} = A_{\alpha}^{\sigma_*} + \mathcal{O}(\epsilon^2)\,. \tag{3.15}$$

3.3 A simple, but interesting RG flow

There is an interesting example of a boundary perturbation for which the theory remains exactly soluble. Let us consider the boundary Liouville theory at $\sigma = 0$. In section 2.6 we had observed that the primary boundary field Φ_0 projects $\mathcal{H}_{0,0}^{\rm B}$ onto the sector $\mathcal{V}_{0,c}$, and $\langle B_0|$ onto $\langle B_{\rm D}|$. These properties will imply that the RG flow generated by the boundary perturbation Φ_0 is almost trivial. Let us consider

$$\begin{aligned} \langle B_{0,\lambda}|V_\alpha(z,\bar z)\rangle &:= \langle B_0|\,{\rm Pexp}\Big(-\frac{\lambda}{a}\int_{\partial\Sigma}dx\,\Phi_0\Big)V_\alpha(z,\bar z)\rangle \\ &= \langle B_0|\exp\Big(-2\pi\lambda\frac{L}{a}\Phi_0\Big)V_\alpha(z,\bar z)\rangle \end{aligned} \tag{3.16}$$

By using $\Phi_0^2 = \Phi_0$ one may further calculate

$$\langle B_{0,\lambda}|V_\alpha(z,\bar z)\rangle = \langle B_0|(1 + (e^{-2\pi\lambda\frac{L}{a}} - 1)\Phi_0)V_\alpha(z,\bar z)\rangle\,. \tag{3.17}$$

The endpoint of the flow is therefore represented by

$$\begin{aligned} \langle B_{0,\infty}|V_\alpha(z,\bar z)\rangle &= \langle B_0|(1-\Phi_0)V_\alpha(z,\bar z)\rangle \\ &= \langle B_b|\,V_\alpha(z,\bar z)\rangle\,, \end{aligned} \tag{3.18}$$

where we have used equations (2.27) and (2.25) to go from the first to the second line. This means that the RG flow generated by the boundary perturbation Φ_0 acts again like a covering transformation: it removes the admixture represented by $\langle B_{\rm D}|$ that is created by performing the analytic continuation w.r.t. $\mu_{\rm B}$ along a closed path starting from $\mu_{\rm B} = -\sqrt{\frac{\mu}{\sin\pi b^2}}\cos\pi b^2$.

4. D-brane decay in two-dimensional string theory

Let us consider D1-branes in noncritical string theories that are characterized by boundary states of the form

$$(B| = \langle B_{\rm N}|_{X^0}\otimes\langle B_\sigma|\,, \tag{4.1}$$

where $\langle B_{\rm N}|_{X^0}$ is supposed to described Neumann type boundary conditions for the X^0-CFT to begin with. We will assume that the reader is familiar with the discussion in section 3 of [4]. Throughout this section we will consider the case of $b = 1 \Leftrightarrow c = 25$.

In the two cases $\sigma \in \mathbb{S}$ and $\sigma \in (1/2, 1)$, the Liouville theory on the strip with boundary conditions labelled by σ on both sides will have a purely continuous spectrum given by (2.7). This implies a purely continuous spectrum of open string tachyons on the D1-branes, which is generated by the on-shell tachyon vertex operators

$$T_E = [e^{i\omega X^0}\Phi^\sigma_\alpha]_{\alpha=1+i\omega}\,. \tag{4.2}$$

The situation changes when σ gets smaller than $1/2$. The bound states in $\mathcal{H}^{\mathrm{B}}_{\sigma_2\sigma_1}$, cf. (2.19), then yield on-shell tachyon vertex operators with imaginary frequencies like

$$T_E = [e^{\nu X^0}\Phi^\sigma_{2\sigma}]_{2\sigma=1-\nu}\,. \tag{4.3}$$

Appearance of imaginary energies usually signals some instability. The very form of (4.3) suggests that a perturbation of the system by (4.3) has an effect that blows up exponentially when time $X^0 \to \infty$.

4.1 Comments on the time-dependent description

Ideally we would like to associate time-dependent solutions of noncritical open string theory to the RG flows discussed in the previous section. The basic idea for a perturbative construction of a time-dependent solution is rather obvious: replace the couplings of the relevant perturbing fields by (yet undetermined) functions of the time coordinate X^0. These functions have to be chosen such that conformal symmetry is preserved by the resulting action. To begin with, let us observe that an ansatz of the simple form

$$\delta S = \kappa \int_{\partial\Sigma} dx\, (e^{\nu X^0}\Phi^\sigma_{1-\nu})(x) \tag{4.4}$$

will not preserve conformal invariance in general. At second order in κ one has to consider

$$\frac{1}{2}\kappa^2 \mathrm{P}\int_{\partial\Sigma} dx_2 dx_1\, (e^{\nu X^0}\Phi^\sigma_\alpha)(x_2)(e^{\nu X^0}\Phi^\sigma_\alpha)(x_1)\,. \tag{4.5}$$

We will again focus on the case that $\Phi^\sigma_\alpha(x)$ is nearly marginal, $0 < \nu^2 \ll 1$. The main contribution to the integral (4.5) then comes from the vicinity of the diagonal $x_2 = x_1$, and may again be estimated by using the operator product expansion. We thereby find a conformally non-invariant contribution of the form

$$\frac{1}{2}\kappa^2\, \frac{c_\nu}{\nu^2} E^\sigma_{\alpha\alpha} \int_{\partial\Sigma} dx\, (e^{2\nu X^0}\Phi_\sigma)(x) + \mathcal{O}(\nu^0)\,, \tag{4.6}$$

where c_ν is constant up to terms of higher order in ν^2. We must therefore modify our ansatz (4.4) by corrections of higher order in κ which will contain the macroscopic boundary field $\Psi_\sigma(x)$ multiplied by $e^{2\nu X^0}$. Note that the first order term (4.4) is the leading one for $X^0 \to -\infty$, however.

One may then try to construct δS in the form

$$\delta S = \kappa \int_{\partial\Sigma} dx\, (e^{\nu X^0}\Phi^\sigma_{1-\nu})(x) + \sum_{n=2}^{\infty} \kappa^n (\delta S)^{(n)}[X^0]\,. \tag{4.7}$$

It seems clear that the higher order corrections in δS are also governed by the OPE of Liouville boundary fields. We had previously observed that the OPE of the boundary fields $\Phi^\sigma_\alpha(x_2)\Phi^\sigma_\alpha(x_1)$ will generically not contain the field $\Phi^\sigma_\alpha(x_1)$ at all (unless $\alpha = 2\sigma$). In this case we would consequently expect that the higher order corrections $(\delta S)^{(n)}$ do not contain $\Phi^\sigma_\alpha(x_1)$ as well. This phenomenon can be seen as a counterpart of the "frozen" couplings that we had encountered in the previous section.

4.2 Unstable vs. "frozen" modes

In either picture we observe an important dichotomy. Perturbations containing microscopic boundary fields generate trivial RG flows/carry purely exponential time dependence, unlike the perturbations which contain macroscopic boundary fields. The crucial difference between these two types of perturbations originates from the different character of the states that are created from the two types of perturbations: the macroscopic boundary fields create *normalizable* states, as opposed to the microscopic boundary fields.

One should observe, however, that the two cases are fundamentally different also from the space-time point of view. The microscopic boundary fields describe open string tachyon configurations which diverge in the weak coupling region $\varphi \to -\infty$, as opposed to the case of macroscopic boundary fields. Only the normalizable tachyon field configurations will appear in quantum fluctuations. This leads us to propose that the perturbations that contain macroscopic boundary fields describe brane decays that can take place spontaneously, whereas perturbations by microscopic boundary fields describe decays triggered by external sources instead. A similar proposal about perturbations by bulk fields was made some years ago by Seiberg and Shenker [23]. Our previous discussion exhibits the world-sheet origin of this phenomenon in a prototypical example.

4.3 A fast decay

The perturbative treatment of time-dependent phenomena is clearly limited to slow processes or to the initial stages of a decay process. In the present case we only know one example where we can go beyond this restriction to study a fast decay of a D-brane.

Let us consider the D-brane described by the boundary state

$$(B_0| = \langle B_{\mathrm{N}}|_{X^0} \otimes \langle B_0| \,, \tag{4.8}$$

which corresponds to the limit $\sigma \to 0$ discussed in section 2.6. The boundary state $(B_0|$ describes a bound state formed by a D1 brane with $\mu_{\mathrm{B,ren}} = \sqrt{\mu_{\mathrm{ren}}}$ and a D0 brane. This bound state is perturbatively unstable as follows from the existence of a normalizable highest weight state within $\mathcal{H}^{\mathrm{B}}_{0,0}$. The corresponding boundary perturbation is obtained by dressing the macroscopic boundary field Φ_0 with e^{X^0},

$$S_{\mathrm{B}} = \lambda \int d\tau\, \Phi_0\, e^{X^0} \,. \tag{4.9}$$

By means of a calculation that is very similar to the one in section 3.3 one finds that the boundary state which describes the decay of the brane characterized by $(B_0|$ is given as

$$(B_{0,\lambda}| = \langle B_{\mathrm{N}}|_{X^0} \otimes \langle B_1| + \langle B_{\mathrm{S},\lambda}|_{X^0} \otimes \langle B_{\mathrm{D}}| \,, \tag{4.10}$$

where $\langle B_{\mathrm{S},\lambda}|_{X^0}$ is the relevant member of the class of boundary states for the X^0-CFT introduced by A. Sen in [11], see e.g. [24, 25, 26] for further results and references. The interpretation of the time-dependent process described by $(B_{0,\lambda}|$ should be clear: the D0 brane decays spontaneously, leaving behind the D1 brane with $\mu_{\mathrm{B,ren}} = \sqrt{\mu_{\mathrm{ren}}}$ as stable remnant.[3]

4.4 Summary: dependence with respect to $\mu_{\mathrm{B,ren}}$

Let us summarize the resulting picture of the dependence of the D1 branes w.r.t. the parameter $\mu_{\mathrm{B,ren}}$ or σ. We will consider the analytic continuation starting from the case that $\sigma \in \mathbb{S}$ which corresponds to $\mu_{\mathrm{B,ren}} > \sqrt{\mu_{\mathrm{ren}}}$. One is then dealing with a D-string that stretches along the Liouville-direction, but gradually disappears in the strong coupling region $\varphi \to \infty$. This follows from the fact that the one-point function $\langle B_\sigma|P\rangle$ decays exponentially as

$$\langle B_\sigma|P\rangle \sim e^{-2\pi P(2\,\mathrm{Re}(\sigma)-1)} \qquad \text{for } P \to \infty\,, \tag{4.11}$$

taking into account that wave-packets with average energies $\bar{E} \sim P$ probe more and more deeply into the region $\varphi \to \infty$ if one increases $\bar{E}$. This means that the effective coupling between closed string wave-packets and the D1 branes with $\sigma < 1/2$ decreases fast with the depth of penetration into the strong coupling region. The boundary cosmological constant $\mu_{\mathrm{B,ren}}$ determines how far the D1 brane extends into the strong coupling region: it stretches further out to $\varphi \to \infty$ if one decreases $\mu_{\mathrm{B,ren}}$.

The properties of the D1 brane change qualitatively as soon as one passes the turning point $\sigma = 1/2 \Rightarrow \mu_{\mathrm{B,ren}} = -\sqrt{\mu_{\mathrm{ren}}}$ upon continuing from $\sigma > 1/2$ to $\sigma < 1/2$: the D1 brane acquires a “mass” near $\varphi \to \infty$. This follows from the exponential growth (4.11) of the one-point function $\langle B_\sigma|P\rangle$ by similar arguments as used in the previous paragraph. The bound state that appears in $\mathcal{H}^{\mathrm{B}}_{\sigma,\sigma}$ for $\sigma < 1/2$ is naturally interpreted as an open string bound to the part of the D1 brane that is localized near $\varphi \to \infty$. These branes, however, are unstable. Near $\sigma = 1/2$ one may use the results from section 3.1 and section 3.2 to conclude that the decay takes place slowly and produces a D1 brane with reduced “mass” near $\varphi \to \infty$.

When σ decreases further we do not expect qualitative changes of the picture as long as we have $\sigma > 0$. At $\sigma = 0$, however, we not only find that the mass near $\varphi \to \infty$ becomes *equal* to the mass of the D0 brane. The results of section 2.6 concerning the decoupling of the sector $\mathcal{V}_{0,c}$ from the rest of the open string spectrum demonstrate that there are no open strings that stretch between the D0 component of the D1 brane with $\sigma = 0$ and the rest. The D0 part of the D1 brane becomes free to decay independently of the rest, as discussed in section 3.3, section 4.3.

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[3]In my talk at Strings 2003 I have made an incorrect statement about this point. I thank J. Maldacena for pointing out that he expects the scenario above to be realized instead.

A. Analytic properties of the structure functions

A.1 Special functions

The function $\Gamma_b(x)$ is a close relative of the double Gamma function studied in [27, 28]. It can be defined by means of the integral representation

$$\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t}\left(\frac{e^{-xt}-e^{-Qt/2}}{(1-e^{-bt})(1-e^{-t/b})} - \frac{(Q-2x)^2}{8e^t} - \frac{Q-2x}{t}\right). \tag{A.1}$$

Important properties of $\Gamma_b(x)$ are

(i) *Functional equation:* $\Gamma_b(x+b) = \sqrt{2\pi} b^{bx-\frac{1}{2}} \Gamma^{-1}(bx)\Gamma_b(x)$.

(ii) *Analyticity:* $\Gamma_b(x)$ is meromorphic, poles: $x = -nb - mb^{-1}$, $n, m \in \mathbb{Z}^{\geq 0}$.

(iii) *Self-duality:* $\Gamma_b(x) = \Gamma_{1/b}(x)$.

The function $S_b(x)$ may be defined by

$$\log S_b(x) = \int_0^\infty \frac{dt}{t}\left(\frac{\sinh t(Q-2x)}{2\sinh bt \sinh b^{-1}t} - \frac{Q-2x}{2t}\right). \tag{A.2}$$

$S_b(x)$ is related to $\Gamma_b(x)$ via

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}. \tag{A.3}$$

The most important properties for our purposes are

(i) *Functional equation:* $S_b(x+b) = 2\sin \pi bx S_b(x)$.

(ii) *Analyticity:* $S_b(x)$ is meromorphic,

$$\begin{aligned} &\text{poles}: \quad x = -(nb + mb^{-1}), \quad n, m \in \mathbb{Z}^{\geq 0}. \\ &\text{zeros}: \quad x = Q + (nb + mb^{-1}), \quad n, m \in \mathbb{Z}^{\geq 0}. \end{aligned} \tag{A.4}$$

(iii) *Asymptotics:*

$$S_b(x) = e^{\mp \frac{\pi i}{2} x(x-Q)} \qquad \text{for } \operatorname{Im}(x) \to \pm\infty. \tag{A.5}$$

(iv) *Inversion relation:* $S_b(x) S_b(Q-x) = 1$.

(v) *Residue:*

$$\operatorname{res}_{x=0} S_b(x) = (2\pi)^{-1}. \tag{A.6}$$

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A.2 Three point functions

To begin with, let us quote the explicit formula for $C^{\sigma_3\sigma_2\sigma_1}_{\alpha_3|\alpha_2\alpha_1}$ found in [18]:

$$\begin{aligned} C^{\sigma_3\sigma_2\sigma_1}_{\alpha_3|\alpha_2\alpha_1} &= (\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{1}{2b}(\alpha_3-\alpha_2-\alpha_1)}\Gamma_b(2Q-\alpha_1-\alpha_2-\alpha_3)\times \\ &\times\frac{\Gamma_b(\alpha_2+\alpha_3-\alpha_1)\Gamma_b(Q+\alpha_2-\alpha_1-\alpha_3)\Gamma_b(Q+\alpha_3-\alpha_1-\alpha_2)}{\Gamma_b(2\alpha_3-Q)\Gamma_b(Q-2\alpha_2)\Gamma_b(Q-2\alpha_1)\Gamma_b(Q)}\times \\ &\times\frac{S_b(\alpha_3+\sigma_1-\sigma_3)S_b(Q+\alpha_3-\sigma_3-\sigma_1)}{S_b(\alpha_2+\sigma_2-\sigma_3)S_b(Q+\alpha_2-\sigma_3-\sigma_2)}\int_{\mathbb{R}+i0}ds\prod_{k=1}^{4}\frac{S_b(U_k+is)}{S_b(V_k+is)}\,, \end{aligned} \tag{A.7}$$

The coefficients U_k, V_k and $k=1,\ldots,4$ are defined as

$$\begin{aligned} U_1 &= \sigma_1+\sigma_2-\alpha_1\,, & V_1 &= \sigma_2+\bar{\sigma}_3-\alpha_1+\alpha_3\,, \\ U_2 &= \bar{\sigma}_1+\sigma_2-\alpha_1\,, & V_2 &= \sigma_2+\bar{\sigma}_3-\alpha_1+\bar{\alpha}_3\,, \\ U_3 &= \alpha_2+\sigma_2-\sigma_3\,, & V_3 &= 2\sigma_2\,, \\ U_4 &= \bar{\alpha}_2+\sigma_2-\sigma_3\,, & V_4 &= Q\,. \end{aligned}$$

We have used the notation $\bar{\sigma}_l:=Q-\sigma_l$, $\bar{\alpha}_l:=Q-\alpha_l$, $l=1,2,3$.

Our aim is to study the analytic properties of the three point function $D^{\sigma_3\sigma_2\sigma_1}_{\alpha_3\alpha_2\alpha_1}$ of the fields $\Phi^{\sigma_2\sigma_1}_{\alpha}(x)$, which can be written in terms of $C^{\sigma_3\sigma_2\sigma_1}_{\alpha_3|\alpha_2\alpha_1}$ as

$$D^{\sigma_3\sigma_2\sigma_1}_{\alpha_3\alpha_2\alpha_1} = g^{\sigma_1\sigma_3}_{\alpha_3}g^{\sigma_3\sigma_2}_{\alpha_2}g^{\sigma_2\sigma_1}_{\alpha_1}C^{\sigma_3\sigma_2\sigma_1}_{Q-\alpha_3|\alpha_2\alpha_1}\,. \tag{A.8}$$

The result is simplest for the three point function $D^{\sigma_3\sigma_2\sigma_1}_{\tilde{\alpha}_3\tilde{\alpha}_2\tilde{\alpha}_1}$ of the fields $\tilde{\Phi}^{\sigma_2\sigma_1}_{\alpha}(x)$, which is related to $D^{\sigma_3\sigma_2\sigma_1}_{\alpha_3\alpha_2\alpha_1}$ via (2.18). We are going to prove the following assertion:

The dependence of $D^{\sigma_3\sigma_2\sigma_1}_{\tilde{\alpha}_3\tilde{\alpha}_2\tilde{\alpha}_1}$ is meromorphic with respect to its six variables $D^{\sigma_3\sigma_2\sigma_1}_{\tilde{\alpha}_3\tilde{\alpha}_2\tilde{\alpha}_1}$ has poles if and only if

$$\sum_{k=1}^{3}\epsilon_k(2\alpha_i-Q)+Q+2(nb+mb^{-1})=0\,, \tag{A.9}$$

where $\epsilon_k\in\{+,-\}$ and $n,m\in\mathbb{Z}^{\geq 0}$.

This set of poles coincides precisely with the set of poles of the three point function $C(\alpha_3,\alpha_2,\alpha_1)$ of the primary field $V_\alpha(z,\bar{z})$ in the bulk. It is remarkable and important for our present aims that the dependence of $D^{\sigma_3\sigma_2\sigma_1}_{\tilde{\alpha}_3\tilde{\alpha}_2\tilde{\alpha}_1}$ w.r.t. the variables σ_k, $k=1,2,3$ is *entire analytic*. To establish our claim (A.9) becomes straightforward once we understand the meromorphic continuation of the integral that appears in (A.7).

A.3 Meromorphic continuation of certain integrals

We are forced to study the dependence of the integral

$$\int_{\mathbb{R}+i0}ds\prod_{k=1}^{4}\frac{S_b(U_k+is)}{S_b(V_k+is)} \tag{A.10}$$

on its parameters. The integrand behaves for large $|s|$ as $e^{\pi Q|s|\sum_{k=1}^4(U_k-V_k)}$. By noting that $\sum_{k=1}^4(U_k-V_k)=-Q$ in our case we may conclude that the convergence of the integral (A.10) does not pose any problems. Let us furthermore note that the integrand of (A.10) has poles at $is = -U_k - nb - mb^{-1}$, $k = 1, \ldots, 4$, $n, m \in \mathbb{Z}^{\geq 0}$, and poles at $is = Q - V_k + nb + mb^{-1}$, $k = 1, \ldots, 4$, $n, m \in \mathbb{Z}^{\geq 0}$, in the right half plane. As long as $0 < \mathrm{Re}(U_k)$ we therefore find all poles at $is = -U_k - nb - mb^{-1}$ strictly in the left half plane and if furthermore $\mathrm{Re}(V_k) < Q$ then all poles at $is = Q - V_k + nb + mb^{-1}$ are localized in the right half plane only. We conclude that the integral (A.10) is analytic w.r.t. U_k and V_k, $k = 1, \ldots, 4$ as long as $0 < \mathrm{Re}(U_k)$ and $\mathrm{Re}(V_k) < Q$ hold for $k = 1, \ldots, 4$.

An analytic continuation of the integral (A.10) to *generic* values of U_k, V_k can be defined by replacing the contour $\mathbb{R} + i0$ in (A.10) by a contour $\mathcal{C}$ that is suitably indented around the strings of poles at $is = -U_k - nb - mb^{-1}$ that have entered the right half-plane, as well as the strings of poles at $is = Q - V_k + nb + mb^{-1}$ that can be found in the left half-plane. Of course one should require that $\mathcal{C}$ approaches the real axis for $|s| \to \infty$. Such a contour will exist iff none of the poles at $is = -U_k - nb - mb^{-1}$, $k = 1, \ldots, 4$, happens to lie on top of a pole at $is = Q - V_l + nb + mb^{-1}$, $l = 1, \ldots, 4$.

Otherwise let us study the behavior of (A.10) when $-U_k - nb - mb^{-1} = Q - V_l + n'b + m'b^{-1} + \epsilon$ with $\epsilon \downarrow 0$. The singular behavior of the integral may be determined by deforming the contour $\mathcal{C}$ into a new contour that is given as the sum of a contour $\mathcal{C}'$ which separates the pole at $is = Q - V_l + nb + mb^{-1}$ from $is = +\infty$ and a small circle around $is = Q - V_l + nb + mb^{-1}$. The factor $S_b(U_k + s)$ in the numerator of the integrand yields a residue proportional to $S_b(\epsilon + (n' - n)b + (m' - m)b^{-1})$ which develops a pole for $\epsilon \to 0$ if $n' - n < 0$ and $m' - m < 0$. This implies that (A.10) has poles if

$$Q + U_k - V_l + nb + mb^{-1} = 0\,, \qquad k, l = 1, \ldots, 4\,, \quad n, m \in \mathbb{Z}^{\geq 0}\,. \tag{A.11}$$

One may finally convince oneself that the contour $\mathcal{C}'$ can always be chosen such that integration over $\mathcal{C}'$ is nonsingular. The list of poles given in (A.11) is therefore complete.

A.4 Special values of the three point functions

The general expression (A.7) simplifies considerably for certain values of the parameters. Of particular relevance for us are the cases where one of α_i, $i = 1, 2, 3$ is set to a value that parametrizes a discrete representation in the spectrum. As an example let us consider the case $\alpha_1 = \sigma_1 + \sigma_2$. The integral in (A.7) becomes singular in this case since the contour of integration becomes pinched between the poles of $S_b(U_1 + s)$ and $S_b(V_4 + s)$. In order to extract the singular part of the integral let us deform the contour into the contour $\mathbb{R} - i0$ plus a small circle around $s = 0$. Only the contribution from the circle around $s = 0$ displays a singular behavior for $U_1 \to 0$, and is given by

$$\prod_{k=1}^{3} \frac{S_b(U_{k+1})}{S_b(V_k)} S_b(U_1)\,. \tag{A.12}$$

Once we multiply $C^{\sigma_3\sigma_2\sigma_1}_{Q-\alpha_3|\alpha_2\alpha_1}$ by the factor $g^{\sigma_2\sigma_1}_{\alpha_1}$, the pole from $S_b(U_1)$ in (A.12) gets cancelled by a zero of $g^{\sigma_2\sigma_1}_{\alpha_1}$, leading to a finite result for $D^{\sigma_3\sigma_2\sigma_1}_{\alpha_3\alpha_2\alpha_1}$. In this way it becomes straightforward to calculate the special values $D^{\sigma\sigma\sigma}_{\alpha_3,\alpha_2,2\sigma}$, $D^{\sigma}_{\alpha} := D^{\sigma\sigma\sigma}_{\alpha,2\sigma,2\sigma}$ and $D_{\sigma} :=$

$D^{\sigma\sigma\sigma}_{2\sigma,2\sigma,2\sigma}$ explicitly.

$$\begin{aligned}
D^{\sigma\sigma\sigma}_{\alpha_3,\alpha_2,2\sigma} &= (\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{Q}{2b}}\frac{\Gamma_b(2\sigma)\Gamma_b^3(2Q-2\sigma)\Gamma_b(Q)}{\Gamma_b(2Q-4\sigma)\Gamma_b(\alpha_2)\Gamma_b(Q-\alpha_2)\Gamma_b(\alpha_3)\Gamma_b(Q-\alpha_3)}\times\\
&\times\frac{\Gamma_b(Q+\alpha_3-\alpha_2-2\sigma)\Gamma_b(Q+\alpha_2-\alpha_3-2\sigma)}{\Gamma_b(Q+\alpha_3-2\sigma)\Gamma_b(Q+\alpha_2-2\sigma)}\times\\
&\times\frac{\Gamma_b(2Q-\alpha_3-\alpha_2-2\sigma)\Gamma_b(\alpha_3+\alpha_2-2\sigma)}{\Gamma_b(2Q-\alpha_3-2\sigma)\Gamma_b(2Q-\alpha_2-2\sigma)}\,, \qquad \text{(A.13)}\\
D^{\sigma}_{\alpha} &= (\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{Q}{2b}}\frac{\Gamma_b^3(2Q-2\sigma)}{\Gamma_b^2(2Q-4\sigma)\Gamma_b(Q-2\sigma)}\times\\
&\times\frac{\Gamma_b(Q+\alpha-4\sigma)\Gamma_b(2Q-\alpha-4\sigma)}{\Gamma_b(Q+\alpha-2\sigma)\Gamma_b(2Q-2\sigma-\alpha)} \qquad \text{(A.14)}
\end{aligned}$$

and

$$D_\sigma = (\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{Q}{2b}}\frac{\Gamma_b^3(2Q-2\sigma)}{\Gamma_b^3(2Q-4\sigma)}\frac{\Gamma_b(2Q-6\sigma)}{\Gamma_b(Q)}\,. \qquad \text{(A.15)}$$

The epxression for the two-point function given in formula (2.22) is obtained from (A.14) by setting $\alpha = 0$. We finally recover the formula (2.24) for the OPE coefficient E_σ via $E_\sigma = D_\sigma/d_\sigma$.

B. The bulk-boundary two-point function

It remains to study the bulk-boundary two-point function

$$A^{\sigma}_{\beta|\alpha} \equiv \langle B_\sigma|\Psi^{\sigma\sigma}_{\beta}(0)V_\alpha(z,\bar{z})|0\rangle_{z=\frac{i}{2}}\,. \qquad \text{(B.1)}$$

The following expression for $A^{\sigma}_{\beta|\alpha}$ was found in [29]:

$$A^{\sigma}_{\beta|\alpha} = N\rho_{\beta|\alpha}\int_{-\infty}^{\infty}dt\prod_{\epsilon=\pm}\frac{S_b(\frac{1}{2}(2\alpha+\beta-Q)+i\epsilon t)}{S_b(\frac{1}{2}(2\alpha-\beta+Q)+i\epsilon t)}e^{2\pi t(2\sigma-Q)}\,. \qquad \text{(B.2)}$$

The prefactor $\rho_{\beta|\alpha}$ in (B.2) is defined by

$$\rho_{\beta|\alpha} = (\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{Q-2\alpha-\beta}{2b}}\frac{\Gamma_b^3(Q-\beta)\Gamma_b(2Q-2\alpha-\beta)\Gamma_b(2\alpha-\beta)}{\Gamma_b(Q)\Gamma_b(\beta)\Gamma_b(Q-2\beta)\Gamma_b(2\alpha)\Gamma_b(Q-2\alpha)}\,. \qquad \text{(B.3)}$$

The precise numerical value of the normalization factor N is crucial in section 3.3. We will therefore begin by describing how to fix the value of N.

B.1 Determination of the normalization factor N

Since $\lim_{\beta\to 0}\Psi^{\sigma_2\sigma_1}_{\beta}(x) = 1$ [18] we have to choose N such that $A^{\sigma}_{\beta|\alpha}$ satisfies

$$\lim_{\beta\to 0}A^{\sigma}_{\beta|\alpha} = A^{\sigma}_{\alpha}\,. \qquad \text{(B.4)}$$

In order to evaluate the limit on the left hand side of (B.4) we need to observe that the integral in (B.2) becomes singular for $\beta \downarrow 0$ due to a pinching of the contour of integration by the following poles:

$$\begin{aligned} \text{UHP}: \quad &(a) \quad 2t = i(2\alpha - Q + \beta)\,, \qquad &(b) \quad 2t = i(Q - 2\alpha + \beta)\,, \\ \text{LHP}: \quad &(c) \quad 2t = i(2\alpha - Q - \beta)\,, \qquad &(d) \quad 2t = i(Q - 2\alpha - \beta)\,. \end{aligned} \tag{B.5}$$

We will deform the contour of integration into a contour that passes above the poles (a) and (b) in the upper half plane, plus two circles around these poles. Taking into account (A.6) and (A.3) then yields the following formula

$$\begin{aligned} \lim_{\beta\to 0} A^{\sigma}_{\beta|\alpha} &= N\left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{Q-2\alpha}{2b}} \frac{\Gamma_b(2Q-2\alpha)}{\Gamma_b(Q-2\alpha)} \frac{2\cos(\pi(2\alpha-Q)(2\sigma-Q))}{S_b(2\alpha)S_b(2Q-2\alpha)} \\ &= N\, v(P) \frac{\cosh(2\pi(2\sigma-Q)P)}{\sinh 2\pi b P \sinh 2\pi b^{-1} P} \end{aligned} \tag{B.6}$$

where we have set $\alpha = \frac{Q}{2} + iP$ and used that

$$v(P) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{Q-2\alpha}{2b}} \frac{\Gamma_b(2Q-2\alpha)}{\Gamma_b(Q-2\alpha)}\,. \tag{B.7}$$

By comparing with (2.6), (2.1) we may now read off that $N = -2^{\frac{5}{4}}\sqrt{\pi}$.

B.2 Analyticity of $\langle B_\sigma |\, \Phi^{\sigma\sigma}_{\beta}(0) V_\alpha(z,\bar{z})\, |\, 0 \rangle_{z=\frac{i}{2}}$ w.r.t. σ

Our next aim is to show that $\langle B_\sigma | \Phi^{\sigma\sigma}_{\beta}(0) V_\alpha(z,\bar{z}) | 0 \rangle_{z=\frac{i}{2}}$ is entire analytic in the variable σ. It is not straightforward to read off the relevant analytic properties of $A^{\sigma}_{\beta|\alpha}$ from (B.2). We will therefore present an alternative integral representation from which the desired information can be read off easily.

To begin with, let us observe that formula (B.2) may be written as

$$N^{-1}\rho^{-1}_{\beta|\alpha} A^{\sigma}_{\beta|\alpha} = e^{-\pi i(2\alpha+\beta-Q)(2\sigma-Q)} \frac{S_b(2\alpha+\beta-Q)S_b(\beta)}{S_b(2\alpha)} F_b(2\alpha+\beta-Q,\beta,2\alpha;Q-2\sigma)\,, \tag{B.8}$$

where $F_b(A,B,C;x)$ is the b-hypergeometric function defined as [30, 31]

$$F_b(A,B,C;x) \equiv \frac{S_b(C)}{S_b(A)S_b(B)} \int_{\mathbb{R}+i0} ds\, \frac{S_b(A+is)S_b(B+is)}{S_b(C+is)S_b(Q+is)} e^{-2\pi x s}\,. \tag{B.9}$$

The following identity was established in [31]:

$$F_b(2\alpha+\beta-Q,\beta,2\alpha;Q-2\sigma) = \frac{G_b(2\alpha)}{G_b(\beta)G_b(2\alpha-\beta)} \Psi_b(2\alpha+\beta-Q,\beta,2\alpha;2Q-2\sigma-\beta)\,, \tag{B.10}$$

where $G_b = e^{\frac{\pi i}{2}(x^2-xQ)}S_b(x)$ and $\Psi_b(A,B,C;y)$ is defined by the integral representation

$$\Psi_b(A,B,C;y) \equiv \int_{\mathbb{R}+i0} ds\, \frac{G_b(y+is)G_b(C-B+is)}{G_b(y+A+is)G_b(Q+is)} e^{-2\pi s B}\,. \tag{B.11}$$

From (B.11) it is straightforward to read off the analytic properties w.r.t. σ in the same way as explained in appendix A.3. We find poles only if $Q - \beta \pm (Q - 2\sigma) = -nb - mb^{-1}$. These poles are cancelled by the multiplication with $g^{\sigma\sigma}_{\beta}$, showing that $g^{\sigma\sigma}_{\beta} A^{\sigma}_{\beta|\alpha}$ is indeed entire analytic in the variable σ.

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B.3 Special values of $\langle B_\sigma|\Phi^{\sigma\sigma}_\beta(0)V_\alpha(z,\bar{z})|0\rangle_{z=\frac{i}{2}}$

Of particular importance for us will be certain special values of

$$\langle B_\sigma|\Phi_\sigma(0)V_\alpha(z,\bar{z})|0\rangle_{z=\frac{i}{2}} = \lim_{\beta\to 2\sigma} g^{\sigma\sigma}_\beta A^\sigma_{\beta|\alpha}\,. \tag{B.12}$$

In order to calculate (B.12) let us begin by considering the behavior of the integral in (B.2) for $\beta\to 2\sigma$. With the help of eq. (A.5) one finds that the integrand has leading asymptotics $e^{2\pi(2\sigma-Q\mp(Q-\beta))t}$ for $t\to\pm\infty$. It follows that the integral in (B.2) behaves near $\beta=2\sigma$ as $1/2\pi(2\sigma-\beta)$, which implies that

$$A^\sigma_{\beta|\alpha}\sim \rho_{\beta|\alpha}\frac{N}{2\pi(2\sigma-\beta)}+(\text{regular})\,. \tag{B.13}$$

Let us furthermore note that (A.3) and (A.6) imply that $g^{\sigma\sigma}_\beta$ vanishes at $\beta=2\sigma$ as

$$g^{\sigma\sigma}_\beta\sim(\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{\sigma}{b}}\frac{\Gamma_b(Q-4\sigma)}{\Gamma_b(2Q-4\sigma)}\frac{\Gamma_b(2\sigma)\Gamma_b(2Q-2\sigma)}{\Gamma_b^2(Q-2\sigma)}\left[2\pi(2\sigma-\beta)\right]. \tag{B.14}$$

By combining (B.13) and (B.14) we are lead to the expression

$$\begin{aligned}\langle B_\sigma|\Phi_\sigma(0)V_\alpha(z,\bar{z})|0\rangle_{z=\frac{i}{2}} &= N(\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{Q-2\alpha}{2b}}\times \\ &\quad\times\frac{\Gamma_b(2Q-2\sigma)\Gamma_b(Q-2\sigma)}{\Gamma_b(2Q-4\sigma)\Gamma_b(Q)}\frac{\Gamma(2\alpha-2\sigma)\Gamma_b(2Q-2\alpha-2\sigma)}{\Gamma(2\alpha)\Gamma(Q-2\alpha)}\,.\end{aligned} \tag{B.15}$$

Equation (3.12) in section 3.3. follows from (B.15) by means of a straightforward calculation.

C. Analytic continuation of the boundary OPE

The operator product expansion of the boundary fields $\Phi^{\sigma_2\sigma_1}_{\alpha_1}(x)$ can be read off from the factorization expansion of a four-point function

$$\langle\,\Phi^{\sigma_1\sigma_4}_{\alpha_4}(x_4)\cdots\Phi^{\sigma_2\sigma_1}_{\alpha_1}(x_1)\rangle\,. \tag{C.1}$$

In the case that $\sigma_k\in\mathbb{S}$ and $\alpha_k\in\mathbb{S}$, $k=1,\dots,4$ we know from [18] and [19], section 5, that the four-point function (C.1) can be represented by an expansion of the following form

$$\langle\Phi^{\sigma_1\sigma_4}_{\alpha_4}(x_4)\cdots\Phi^{\sigma_2\sigma_1}_{\alpha_1}(x_1)\rangle=\int_{\mathbb{S}}d\alpha\, m^{\sigma_3\sigma_1}_\alpha D^{\sigma_4\sigma_3\sigma_1}_{\alpha_4\alpha_3\alpha}D^{\sigma_3\sigma_2\sigma_1}_{\alpha\alpha_2\alpha_1}\mathcal{F}_\alpha\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](x_4,\dots,x_1)\,. \tag{C.2}$$

The measure $m^{\sigma_2\sigma_1}_\alpha$ represents a natural analog of the Plancherel measure for the boundary Liouville theory and is given by

$$m^{\sigma_2\sigma_1}_\alpha=\frac{1}{g^{\sigma_2\sigma_1}_\alpha g^{\sigma_1\sigma_2}_{Q-\alpha}}=\mathrm{m}(\alpha)\frac{1}{2\pi}D(\sigma_2,\alpha,\sigma_1)\,, \tag{C.3}$$

where $D(\alpha_3,\alpha_2,\alpha_1)$ is the three-point function of the bulk fields $W_\alpha(z,\bar{z}):=\nu_\alpha V_\alpha(z,\bar{z})$, with $\nu_\alpha\equiv v(P)$ (see (2.2)) for $\alpha=\frac{Q}{2}+iP$, and $\mathrm{m}(\alpha)=4\sin\pi b(2\alpha-Q)\sin\pi b(Q-2\alpha)$.

In order to disentangle the two main effects it is useful to focus on the following two particular cases.

C.1 Continuation w.r.t. the parameters σ

Let us first consider the analytic continuation w.r.t. the parameters σ_k, $k=1,\dots,4$ while keeping the variables α_k, $k=1,\dots,4$ within the domain

$$
\begin{aligned}
&|\,\mathrm{Re}(\alpha_1-\alpha_2)|<\frac{Q}{2}\,, \qquad &&|\,\mathrm{Re}(\alpha_1+\alpha_2-Q)|<\frac{Q}{2}\,,\\
&|\,\mathrm{Re}(\alpha_3-\alpha_4)|<\frac{Q}{2}\,, \qquad &&|\,\mathrm{Re}(\alpha_3+\alpha_4-Q)|<\frac{Q}{2}
\end{aligned}
\tag{C.4}
$$

The results of appendix A.2. and [14, section 7.1], imply that the integrand is meromorphic in the range under consideration, with poles only coming from the measure $m_\alpha^{\sigma_2\sigma_1}$. The discussion of the meromorphic continuation then proceeds along rather similar lines as in the appendix A.3, leading to the conclusion that (C.2) is replaced by an expression of the following form:

$$
\begin{aligned}
\langle \Phi_{\alpha_4}^{\sigma_1\sigma_4}(x_4)\cdots\Phi_{\alpha_1}^{\sigma_2\sigma_1}(x_1)\rangle = &\int_{\mathbb{S}} d\alpha\, m_\alpha^{\sigma_3\sigma_1} D_{\alpha_4\alpha_3\alpha}^{\sigma_4\sigma_3\sigma_1} D_{\alpha\alpha_2\alpha_1}^{\sigma_3\sigma_2\sigma_1}\mathcal{F}_\alpha\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](x_4,\dots,x_1)+ \\
&+\sum_{\alpha\in\mathbb{D}_{\sigma_3\sigma_1}}\frac{1}{\|w_{\alpha;\sigma_3,\sigma_1}\|^2} D_{\alpha_4\alpha_3\alpha}^{\sigma_4\sigma_3\sigma_1} D_{\alpha\alpha_2\alpha_1}^{\sigma_3\sigma_2\sigma_1}\mathcal{F}_\alpha\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](x_4,\dots,x_1)\,.
\end{aligned}
\tag{C.5}
$$

The terms that are summed in the third line of (C.5) are produced by poles of $m_\alpha^{\sigma_3\sigma_1}$ that cross the contour of integration when one continues w.r.t. the variables σ_1 and σ_3. The summation is extended over the set

$$
\mathbb{D}_{\sigma_3\sigma_1}=\Big\{\alpha\in\mathbb{C};\ \alpha=Q-\sigma_{st}+nb+mb^{-1}<\frac{Q}{2}\,, n,m\in\mathbb{Z}^{\geq 0};\ s,t\in\{+,-\}\Big\}, \tag{C.6}
$$

where $\sigma_{st}=s(\sigma_1-\frac{Q}{2})+t(\sigma_3-\frac{Q}{2})$. The form of the factorization (C.5) is easy to understand: the representations $\mathcal{V}_{\alpha,c}$ with $\alpha\in\mathbb{D}_{\sigma_3\sigma_1}$ generate the discrete part in the spectrum $\mathcal{H}^{\mathrm{B}}_{\sigma_3\sigma_1}$ [16]. The factorization (C.5) is therefore just the result of inserting a complete set of intermediate states between $\Phi_{\alpha_3}^{\sigma_4\sigma_3}(x_3)$ and $\Phi_{\alpha_2}^{\sigma_3\sigma_2}(x_2)$, taking into account that $\Phi_{\alpha_2}^{\sigma_3\sigma_2}(x_2)\Phi_{\alpha_1}^{\sigma_2\sigma_1}(x_1)$ creates states within $\mathcal{H}^{\mathrm{B}}_{\sigma_3\sigma_1}$ when acting on the vacuum.

C.2 Continuation w.r.t. the parameters α

Let us now consider the analytic continuation w.r.t. the parameters α_k, $k=1,\dots,4$ while keeping the variables σ_k, $k=1,\dots,4$ within the domain

$$
|\,\mathrm{Re}(\sigma_1-\sigma_3)|<\frac{Q}{2}\,, \qquad |\,\mathrm{Re}(\sigma_1+\sigma_3-Q)|<\frac{Q}{2}\,. \tag{C.7}
$$

We now only have to consider the poles of the three-point functions $D_{\alpha_4\alpha_3\alpha}^{\sigma_4\sigma_3\sigma_1}$ and $D_{\alpha\alpha_2\alpha_1}^{\sigma_3\sigma_2\sigma_1}$. The discussion is largely parallel to [14, section 7.1], allowing us to conclude that the for generic values of α_k, $k=1,\dots,4$ we get a factorization of the following form:

$$
\begin{aligned}
\langle\, \Phi_{\alpha_4}^{\sigma_1\sigma_4}(x_4)\dots\Phi_{\alpha_1}^{\sigma_2\sigma_1}(x_1)\,\rangle = &\int_{\mathbb{S}} d\alpha\, m_\alpha^{\sigma_3\sigma_1} D_{\alpha_4\alpha_3\alpha}^{\sigma_4\sigma_3\sigma_1} D_{\alpha\alpha_2\alpha_1}^{\sigma_3\sigma_2\sigma_1}\mathcal{F}_\alpha\big[\begin{smallmatrix}\alpha_3 \alpha_2\\ \alpha_4 \alpha_1\end{smallmatrix}\big](x_4,\dots,x_1)+ \\
&+\sum_{\alpha\in\mathbb{D}_{\alpha_2\alpha_1}} m_\alpha^{\sigma_3\sigma_1} D_{\alpha_4\alpha_3\alpha}^{\sigma_4\sigma_3\sigma_1} d_{\alpha\alpha_2\alpha_1}^{\sigma_3\sigma_2\sigma_1}\mathcal{F}_\alpha\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](x_4,\dots,x_1)+ \\
&+\sum_{\alpha\in\mathbb{D}_{\alpha_3\alpha_4}} m_\alpha^{\sigma_3\sigma_1} d_{\alpha_4\alpha_3\alpha}^{\sigma_4\sigma_3\sigma_1} D_{\alpha\alpha_2\alpha_1}^{\sigma_3\sigma_2\sigma_1}\mathcal{F}_\alpha\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](x_4,\dots,x_1)\,,
\end{aligned}
\tag{C.8}
$$

where $d^{\sigma_3\sigma_2\sigma_1}_{\alpha_3\alpha_2\alpha_1}$ is the relevant residue of $D^{\sigma_3\sigma_2\sigma_1}_{\alpha_3\alpha_2\alpha_1}$,

$$d^{\sigma_3\sigma_2\sigma_1}_{\alpha_3\alpha_2\alpha_1} = 2\pi i \operatorname*{Res}_{\alpha_3\in\mathbb{D}_{\alpha_2\alpha_1}} D^{\sigma_3\sigma_2\sigma_1}_{\alpha_3\alpha_2\alpha_1}\,. \tag{C.9}$$

In the general case one clearly finds discrete terms of the two different types appearing in (C.5) and (C.8) respectively.

D. The limit $\sigma \to 0$

In this appendix we present the proofs of the statements presented in section 2.6.

D.1 Proof of (2.26)

To begin with, let us note that the coefficient $G^{\sigma_2\sigma_1}_{\alpha}$ is indeed finite in the limit $\sigma_i \to 0$, $i = 1,2$ for generic values of α, unlike $g^{\sigma_2\sigma_1}_{\alpha}$. It follows that expectation values involving the boundary fields $\tilde{\Phi}^{\sigma_2\sigma_1}_{\alpha}(x)$ will be generically finite. The boundary fields $\Phi_\sigma(x)$ which create the bound state with lowest conformal dimension do not need to be renormalized before taking $\sigma \to 0$. This can be seen from our expression (2.22) for the two-point function of the fields $\Phi_\sigma(x)$. Let us therefore introduce the notation

$$\tilde{\Phi}_\alpha(x) := \lim_{\sigma\to 0}\tilde{\Phi}^{\sigma\sigma}_{\alpha}(x)\,, \qquad \Phi_0(x) := \lim_{\sigma\to 0}\Phi_\sigma(x) \tag{D.1}$$

for the relevant boundary fields in the $\sigma = 0$ boundary Liouville theory as well as

$$\begin{aligned}
D_{\tilde{\alpha}_3\tilde{\alpha}_2\tilde{\alpha}_1} &:= \lim_{\sigma\to 0}\langle\tilde{\Phi}^{\sigma\sigma}_{\alpha_3}(\infty)\tilde{\Phi}^{\sigma\sigma}_{\alpha_2}(1)\tilde{\Phi}^{\sigma\sigma}_{\alpha_1}(0)\rangle\\
D_{\tilde{\alpha}_3\tilde{\alpha}_2|0} &:= \lim_{\sigma\to 0}\langle\tilde{\Phi}^{\sigma\sigma}_{\alpha_3}(\infty)\tilde{\Phi}^{\sigma\sigma}_{\alpha_2}(1)\Phi^{\sigma\sigma}_{2\sigma}(0)\rangle\\
D_{\tilde{\alpha}_3|0,0} &:= \lim_{\sigma\to 0}\langle\tilde{\Phi}^{\sigma\sigma}_{\alpha_3}(\infty)\Phi^{\sigma\sigma}_{2\sigma}(1)\Phi_\sigma(0)\rangle\\
D_{0,0,0} &:= \lim_{\sigma\to 0}\langle\Phi^{\sigma\sigma}_{2\sigma}(\infty)\Phi^{\sigma\sigma}_{2\sigma}(1)\Phi^{\sigma\sigma}_{2\sigma}(0)\rangle
\end{aligned} \tag{D.2}$$

for the corresponding three point functions. By using (A.13) and (A.14) one may verify that

$$D_{\tilde{\alpha}_3\tilde{\alpha}_2|0} = 0\,, \qquad \text{and} \qquad D_{\tilde{\alpha}_3|0,0} = 0\,, \tag{D.3}$$

whereas $D_{\tilde{\alpha}_3\tilde{\alpha}_2\tilde{\alpha}_1}$ and $D_{0,0,0}$ will be finite. We furthermore need to discuss the structure of the OPE for $\sigma \to 0$. This may be read off from (C.5) if one takes into account the renormalization relating the boundary fields $\tilde{\Phi}^{\sigma_2\sigma_1}_{\alpha}(x)$ and $\Phi^{\sigma_2\sigma_1}_{\alpha}(x)$, equations (2.17) and (2.18), as well as equation (D.3) expressing the vanishing of mixed three point functions. We thereby infer that the operator product expansion of $\tilde{\Phi}_{\alpha_2}(x_2)\tilde{\Phi}_{\alpha_1}(x_1)$ does *not* contain $\Phi_0(x_1)$. By taking into account that the conformal block $\mathcal{F}_\alpha$ has a first order pole at $\alpha = 0$ one may furthermore observe that four-point functions which contain $\tilde{\Phi}_{\alpha_2}(x_2)\Phi_0(x_1)$ vanish. This, together with (D.3) is good enough to infer the vanishing of all mixed correlation functions of only boundary fields.

To conclude, let us show that $\Phi_0(x_1)$ indeed represents the projector onto the sector $\mathcal{V}_{0,c}$. Let us first note that $\Phi_0(x)$ is a primary field with conformal dimension 0. Vanishing of all mixed correlation functions implies that the fusion rules for decoupling of the null vector in $\mathcal{V}_{0,c}$ are satisfied, so that $\partial_x \Phi_0(x) = 0$. Let us finally observe that by comparing (2.22) and (A.15) in the limit $\sigma \to 0$ one finds that

$$\langle \Phi_0 \Phi_0 \Phi_0 \rangle = \langle \Phi_0 \Phi_0 \rangle \,. \tag{D.4}$$

We conclude that $\Phi_0^2 = \Phi_0$, which means that Φ_0 has the correct normalization for representing the projection onto $\mathcal{V}_{0,c}$.

D.2 Proof of (2.27)

By using the operator product expansion of the bulk fields V_α one may reduce the proof of formula (2.27) to the following result:

$$\lim_{\sigma \to 0} \lim_{\beta \to 2\sigma} \langle B_\sigma | \Psi_\beta^{\sigma\sigma}(0) V_\alpha(z,\bar{z}) | 0 \rangle_{z=\frac{i}{2}} = \langle B_{\rm D} | V_\alpha(z,\bar{z}) | 0 \rangle_{z=\frac{i}{2}} \tag{D.5}$$

if $\alpha = \frac{Q}{2} + iP$. Equation (D.5) follows straightforwardly from (B.15), taking into account that $N = -2^{\frac{5}{4}}\sqrt{\pi}$ and that $\langle B_{\rm D} | V_\alpha(z,\bar{z}) | 0 \rangle_{z=\frac{i}{2}} = \sqrt{2\pi}\Psi_{\rm ZZ}(P)$.

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